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To the Mathematical Description of the Rheologic Properties of  
Real Materials

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I Introduction

The laws of deformation of solids as well as the classification of their basic types were already defined in the first half of the last century. A deeper understanding of these laws has then been closely linked with technical utilization or processing of certain concrete materials which are today subdivided according to their deformation properties in classes of real materials.

Most extensive elaboration has been accorded to the theory of elasticity expressing the hitherto most exploited mechanical property common to almost all solids. The theory of plasticity the elaboration of which has been for the most part associated with structural utilization and processing of metals and which has been finding new applications even in the domains of other kinds of materials, ranks second in this respect.

A certain viscosity may be quoted as another mechanical property of solids. Recent development of the relevant theory has been predominantly linked with partial problems relating to the deformation of various materials, such as plastic flow of concrete or creep of metals at elevated temperatures. The theory of viscous liquids has been subjected to far more intensive theoretical treatment.

The universal and versatile application of new synthetic (me-

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macromolecular) substances as well as the need to unify the existing findings on the behaviour of various materials demand a deeper knowledge of the mechanical behaviour of those materials. The principal point in question is the theory of viscoelasticity which enables us to solve a great number of problems relating to their use. From the point of view of mechanics, there exist two aspects of such problems, the physical and the mathematical.

The physical aspect is in the theory of deformation properties of viscoelastic substances intricately connected with the general theory of relaxation phenomena which forms part of the thermodynamics of irreversible processes / 1 / , / 2 / . From this point of view the mechanical properties are described by the equations of state of microprocesses, but are not- barring exceptions- applicable to the solution of deformation states in bodies. Far more frequent is the phenomenologic approach when one deduces from deformation characteristics of simple states, the equations of state of idealized substances which better satisfy the mathematical methods employed in the solution of viscoelastic problems / 3 / , / 4 / .

The present study deals with an analysis of the relations that exist between the deformation characteristics of real materials and the equations of state of idealized substances.

## II Linear relationships

The mutual action between the solid and the liquid phase is considered in mechanical models the essence of relaxation processes taking place in a continuous quasi-homogeneous solid medium. From the classical theory of elasticity we take over the assumption of perfect elasticity of the solid phase, from the classical theory of viscous flow, that of incompressibility of the liquid

viscoelastic substances is that the equations of state express the relationships between deviators of stress and strain, rates of deviators of stress and strain, etc. The linearity of the equations is then conditioned by approximation

$$\frac{d\tilde{e}_{ij}}{dt} \approx \dot{\tilde{e}}_{ij} \quad (\text{II.1})$$

allowable solely for slow changes in stress and strain-low rates.

A description of the rheologic properties of idealized substances known under the name of Voigt and Maxwell, starts from assumptions of that kind. The homogeneous relaxation of deformation is modelled by Voigt's substance /4/ , p. 89 and described by equation

$$S_{ij} = 2G(e_{ij} + \tau_r \dot{e}_{ij}), \quad (\text{II.2})$$

where  $S_{ij}$  is the deviator of stress,  $e_{ij}$  - the deviator of strain,  $\dot{e}_{ij}$  - the deviator of strain rate,  $\tau_r$  - the time of relaxation.

The homogeneous relaxation of stress is modelled by Maxwell's substance /4/ , p.90 and described by equation

$$2G\dot{e}_{ij} = \dot{S}_{ij} + \frac{1}{\tau_M} S_{ij}, \quad (\text{II.3})$$

where  $\dot{S}_{ij}$  is the deviator of stress rate,  $\tau_M$  - the time of stress relaxation. A more general case of such relaxation models is the so-called "standard substance" described by Zener /5/ with the aid of equation

$$2G(e_{ij} + \tau_M \dot{e}_{ij}) = S_{ij} + \tau \dot{S}_{ij}, \quad (\text{II.4})$$

which models a substance with homogeneous inelasticity. A contribution to further extension and deeper elaboration of the

and Alfrey who introduced the following equation for expressing the deformation of viscoelastic idealized substances /4/, p.95

$$2\zeta Q e_{ij} = P s_{ij} \quad (II.5)$$

in which P and Q are linear differential operators

$$P = a_0 + a_1 \frac{\partial}{\partial t} + a_2 \frac{\partial^2}{\partial t^2} + \dots + a_m \frac{\partial^m}{\partial t^m} \quad (II.6)$$

$$Q = b_0 + b_1 \frac{\partial}{\partial t} + b_2 \frac{\partial^2}{\partial t^2} + \dots + b_m \frac{\partial^m}{\partial t^m}$$

Constants  $a_1, b_1$  are functions of the times of relaxation and shear moduli of elasticity.

### III On a class of rheologic models of viscoelastic substances

The above stated linear relationships that are a description of ideal rheologic substances do not imply, however, the way in which they relate to the rheologic behaviour of real materials. In an endeavour to solve this problem we have used for the purpose of analysis of these relations, Taylor's series which is frequently employed for expressing experimental curves in the form of polynomials. The series can also be applied in the sense of physical interpretation as is well known from its frequent use in mechanics. For relation  $\sigma = f(\epsilon)$  (Fig.1) describing elastic behaviour of substances, we obtain through expansion in Taylor's series, expression

$$\sigma \approx f(0) + f'(0)\epsilon + \frac{1}{2!} f''(0)\epsilon^2 + \dots + \frac{1}{n!} f^{(n)}(0)\epsilon^n \quad (III.1)$$

the first term of which gives the initial inhomogeneity of internal structural stresses, and the second - linear - term represents Hooke's law. Difficulties attending solutions of

series. A similar relation  $\dot{\gamma} = g(\tau)$  (Fig.2) holds for steady flow of viscous liquid; by expanding it in Taylor's series we obtain

$$\dot{\gamma} = g(0) + g'(0)\tau + \frac{1}{2!}g''(0)\tau^2 + \dots + \frac{1}{n!}g^{(n)}(0)\tau^n. \quad (\text{III.2})$$

The first term of this expansion again represents the initial inhomogeneity of local velocity, the second expresses Newton's law in which  $g'(0)$  has the meaning of the coefficient of viscosity. The possibility of using terms of higher order is equally limited as in the case of nonlinear elasticity; such terms may be used but sporadically.

This method of determining the idealized phenomena is advantageous in that we know the qualitative approximation against reality; hence in the above quoted cases the analysis directly implies that the validity of Hooke's law is limited by the condition of  $\epsilon \ll 1$ , while condition  $\tau \ll 1$  holds for Newton's law. The approximation is also conditioned by the remainder of Taylor's series which is small if the properties of a real substance are close to the idealized ones.

In our analysis of the rheologic properties we shall use Taylor's series for the purpose of decomposing a composite function, and similarly as in the preceding examples shall consider this expansion a decomposition of a composite rheologic process in simple processes which again represent some idealized states. In the discussion that follows we shall always start from uniaxial states of stress and strain and shall not consider tensor relationships.

a) The equation of state of creep flow is expressed generally by relation

$$\sigma = F[\epsilon(t)], \quad (\text{III.3})$$

which associates with the given value of external stress deformation  $\epsilon$  which changes relative to time in a manner that the relation between  $\sigma$  and  $\epsilon$  in a certain time interval measured from the beginning of stress action  $t_0 = 0$ , is infinitely multivalued. In case of steady relaxation the length of the interval is given by relaxation time  $\tau_r$ . Characteristic curves of steady and unsteady relaxation are shown in Fig. 5.

We shall expand the right-hand side of equation (III.3) in Taylor's series for  $t = 0$ . The expansion yields for expression

$$F[\epsilon(t)] = F_{\epsilon(0)} + \frac{dF_{\epsilon(0)}}{dt} t + \frac{1}{2!} \frac{d^2 F_{\epsilon(0)}}{dt^2} t^2 + \dots + \frac{1}{n!} \frac{d^n F_{\epsilon(0)}}{dt^n} t^n \quad (\text{III.4})$$

and for the differentials of the composite function we obtain

$$\frac{dF_{\epsilon}}{dt} = \frac{dF_{\epsilon}}{d\epsilon} \frac{d\epsilon}{dt},$$

$$\frac{d^2 F_{\epsilon}}{dt^2} = \frac{d^2 F_{\epsilon}}{d\epsilon^2} \left(\frac{d\epsilon}{dt}\right)^2 + \frac{dF_{\epsilon}}{d\epsilon} \frac{d^2 \epsilon}{dt^2}, \quad (\text{III.5})$$

$$\frac{d^3 F_{\epsilon}}{dt^3} = \frac{d^3 F_{\epsilon}}{d\epsilon^3} \left(\frac{d\epsilon}{dt}\right)^3 + 3 \frac{d^2 F_{\epsilon}}{d\epsilon^2} \frac{d\epsilon}{dt} \frac{d^2 \epsilon}{dt^2} + \frac{dF_{\epsilon}}{d\epsilon} \frac{d^3 \epsilon}{dt^3}.$$

On substituting (III.5) in (III.4) we obtain equation (III.3)

in the following form with notation  $\frac{d^n F_{\epsilon}}{d\epsilon^n} = F_{\epsilon}^{(n)}$

$$\begin{aligned} \sigma = & F_{\epsilon} + F_{\epsilon} \left[ \left( \frac{d\epsilon}{dt} \right)_{t=0} t + \frac{1}{2!} \left( \frac{d^2\epsilon}{dt^2} \right)_{t=0} t^2 + \dots + \frac{1}{n!} \left( \frac{d^n\epsilon}{dt^n} \right)_{t=0} t^n \right] + \\ & + F_{\epsilon} \left[ \frac{1}{2!} \left( \frac{d\epsilon}{dt} \right)_{t=0}^2 t^2 + \frac{1}{2!} \left( \frac{d\epsilon}{dt} \frac{d^2\epsilon}{dt^2} \right)_{t=0} t^3 + \dots \right] + \quad (\text{III.6}) \\ & + F_{\epsilon} \left[ \frac{1}{3!} \left( \frac{d\epsilon}{dt} \right)_{t=0}^3 t^3 + \dots \right] + \dots + F_{\epsilon} \left[ \frac{1}{n!} \left( \frac{d\epsilon}{dt} \right)_{t=0}^n t^n + \dots \right]. \end{aligned}$$

This method would be used for decomposition of an experimentally determined relationship. If we wish to carry out decomposition of a physical process, we must exclude the implicitly expressed time from the equations. Since time appears in the  $n$ th power in these expressions, it is necessary to introduce  $g$  assumptions that would imply  $g$  equations permitting us to exclude the implicitly expressed time from equation (III.6). Using for that purpose the assumption of low rates of strain, we can write

$$\left( \frac{d\epsilon}{dt} \right)_{t=0} t = \dot{\epsilon}(t) \tau_1 \quad (\text{III.7})$$

where  $\tau_1$  denotes the relaxation time hitherto not analyzed as to whether  $\tau_1$  or  $\tau_u$  is involved. We shall now extend the assumption agreed on for the rates of strain, to higher time increments of strain, introduce the following relations for all subsequent terms of equation (III.6)

$$\left( \frac{d^m\epsilon}{dt^m} \right)_{t=0} \frac{t^m}{m!} = \dot{\epsilon}^{(m)}(t) \tau_m^m \quad (\text{III.8})$$

and introduce them in equation (III.6) which then gives

$$\begin{aligned} \sigma = & F_E + F_E' (\tau_1 \dot{\epsilon} + \tau_1^2 \ddot{\epsilon} + \tau_1^3 \dddot{\epsilon} + \dots + \tau_n^{(n)} \epsilon^{(n)}) + \\ & + F_E'' (\tau_1^2 \dot{\epsilon}^2 + \tau_1 \tau_2^2 \dot{\epsilon} \ddot{\epsilon} + \dots) + \\ & + F_E''' (\tau_1^3 \dot{\epsilon}^3 + \dots) + \dots + F_E^{(n)} (\tau_1^n \dot{\epsilon}^n + \dots). \end{aligned} \quad (\text{III.9})$$

Equation (III.9) describes already the model of rheologic relationships (III.5), and conditions stated by equations (III.7) and (III.8) are the conditions of similarity. Moreover, eq. (III.9) retains the function for elastic deformation properties and through its intermediary the mutual connection between physical and rheologic quantities in the relaxation deformation process becomes clearly manifest. Linear rheologic quantities are linked with the tangential elastic modulus, rheologic quantities of the second order with the curvature of the deformation characteristic. In order to obtain a linear rheologic model, in equation (III.9) must be linear, which means a perfectly elastic material. The conditions of a linear model are thus further extended in the sense of the decomposition given by equation (III.1); hence it must hold that

$$F_E' = \text{konst.}, F_E'' = F_E''' = F_E^{(iv)} = \dots = F_E^{(n)} = 0 \quad (\text{III.10})$$

or  $\epsilon$  must be close to zero, i.e. the assumption of small deformations must be fulfilled. Under such conditions we obtain for a general linear relaxation model of creep flow the following equation

$$\sigma = F_{\epsilon=0}' (\epsilon + \tau_1 \dot{\epsilon} + \tau_2 \ddot{\epsilon} + \dots + \tau_n^{(n)} \epsilon^{(n)}) \quad (\text{III.11})$$



where  $F'_{\epsilon=0}$  has the meaning of the modulus of elasticity, and  $\tau_m$  that of the relaxation times corresponding to higher states of viscous flow. Multiples  $F'_{\epsilon=0} \tau_m^m$  give the coefficients of viscosity pertinent to those states. If kinematic components higher than the rate of strain

$$\ddot{\epsilon} = \ddot{\epsilon} = \dots = \epsilon^{(n)} = 0, \quad (\text{III.12})$$

do not come into effect in the relaxation process, we obtain from (III.11) the well-known equation of Voigt's model of a viscoelastic substance

$$\sigma = E(\epsilon + \tau \dot{\epsilon}). \quad (\text{III.13})$$

b) In the sections that follow we shall study the behaviour of a viscoelastic liquid described by equation of state

$$\dot{\epsilon} = G[\sigma(t)], \quad (\text{III.14})$$

which is a generalization of (III.2) if in the flow effect of time is manifested in the same manner as in the case of solid the deformation behaviour of which we described by equations (III.1) and (III.3). Irrespective of the kind of strain, we shall denote the strain quantities by  $\epsilon$ , and similarly, stresses by  $\sigma$ . Equation (III.14) (Fig.4) describes the properties of liquids which for a given external velocity  $\dot{\epsilon}$  exhibit in a certain time interval measured from the beginning of steady flow, a variable value of internal stress  $\sigma(t)$  so that the relation between  $\epsilon$  and  $\sigma$  is infinitely multi-valued within the interval of  $(t_0, t)$ . The time in the course of which at a steady flow the value of  $\tau_r$  internal stress also becomes steady, is the time of relaxation. If  $\dot{\epsilon}$  exceeds

hence of flow ensues. The equation of real viscoelastic liquids is evolved in the same manner as that of creep flow. Consequently we shall carry out decomposition of the right-hand side of equation (III.14) without derivation. On the basis of (III.6) we obtain

$$\begin{aligned} \dot{\epsilon} = & G_0 + G_0' \left[ \left( \frac{d\sigma}{dt} \right)_{t=0} t + \frac{1}{2!} \left( \frac{d^2\sigma}{dt^2} \right)_{t=0} t^2 + \dots + \frac{1}{n!} \left( \frac{d^n\sigma}{dt^n} \right)_{t=0} t^n \right] + \\ & + G_0'' \left[ \frac{1}{2!} \left( \frac{d\sigma}{dt} \right)_{t=1}^2 t^2 + \frac{1}{2!} \left( \frac{d\sigma}{dt} \frac{d^2\sigma}{dt^2} \right)_{t=1} t^3 + \dots \right] \quad \text{[III.15]} \\ & + G_0''' \left[ \frac{1}{3!} \left( \frac{d\sigma}{dt} \right)_{t=1}^3 t^3 + \dots \right] + \dots + G_0^{(n)} \left[ \frac{1}{n!} \left( \frac{d\sigma}{dt} \right)_{t=1}^n t^n + \dots \right]. \end{aligned}$$

Analogously to (III.8) we shall introduce assumption

$$\left( \frac{d^m \sigma}{dt^m} \right)_{t=0} \frac{t^m}{m!} = \sigma(t) \tau_m^{(m)} \quad \text{(III.16)}$$

and on introducing (III.16) in (III.8) obtain

$$\begin{aligned} \dot{\epsilon} = & G_0 + G_0' (\tau_1 \dot{\sigma} + \tau_2^2 \ddot{\sigma} + \tau_3^3 \overset{3}{\sigma} + \dots + \tau_n^n \overset{(n)}{\sigma}) + \\ & + G_0'' (\tau_1^2 \overset{2}{\sigma} + \tau_1 \tau_2^2 \overset{2}{\sigma} \overset{2}{\sigma} + \dots) + \quad \text{(III.17)} \\ & + G_0''' (\tau_1^3 \overset{3}{\sigma} + \dots) + \dots + G_0^{(n)} (\tau_1^n \overset{(n)}{\sigma} + \dots). \end{aligned}$$

Equation (III.17) again describes the rheologic behaviour of real viscoelastic liquids, and the mutual connection between quantities  $G_0$  characteristic of the viscous properties of the liquid and the kinematic quantities  $\overset{(n)}{\sigma}$  characterizing the rheologic properties of the liquid become again manifest in it.

A linear rheologic model is obtained on the assumption that

to (III.2) for  $\dot{\epsilon}_0$  close to zero, i.e. the assumption of low rates of strain must be fulfilled. Under such conditions the following equation holds for a linear rheologic model of visco-elastic liquid

$$\dot{\epsilon} = G'_{G=0} \left( \sigma + \tau_1 \dot{\sigma} + \frac{\tau_2^2}{2} \ddot{\sigma} + \dots + \tau_n^n \sigma^{(n)} \right) \quad (\text{III.18})$$

where  $G'_{G=0}$  has the meaning of the reciprocal of the coefficient of viscosity, and  $\tau$  are the relaxation times corresponding to the higher states of elastic deformation. Multiples  $G'_{G=0} \tau^n$  then express the moduli of elasticity of such elastic strain states. If the effect of kinematic components higher than the rate of stress

$$\sigma'' = \sigma''' = \dots = \sigma^{(n)} = 0, \quad (\text{III.19})$$

does not come into play in the relaxation process, we obtain from (III.18) the equation of Maxwell's model of a viscoelastic liquid

$$\dot{\epsilon} = \frac{1}{\eta} (\sigma + \tau_1 \dot{\sigma}), \quad (\text{III.20})$$

e) Substances that display the effect of previous stress and strain in the process of deformation, require further extension of the theory of viscoelasticity. From the point of view of the essence of the phenomenon, the substances in question have properties of both solids - one can make a test for creep(a) as well as of liquids - one can test them for time variations of the viscosity (b). The equation of state of such a substance can therefore be expressed either as the equation of solid (III.1

that a dependence of stress  $\bar{\sigma}$  and strain  $\bar{\epsilon}$  on the history of stress  $\bar{\sigma}$  and strain  $\bar{\epsilon}$  is introduced in them.

Stress  $\bar{\sigma}(t)$  and strain  $\bar{\epsilon}(t)$  are generally obtained through transformation of equations (III.5) and (III.14), or for linear cases, through solution of equations (III.11) and (III.18). If Green's functions are used in the transformation or solution, the stresses and strains are given by expressions

$$\bar{\epsilon}(t) = \int_0^t F[\bar{\epsilon}(t-\nu)] \bar{\sigma}'(\nu) d\nu, \quad (\text{III.21})$$

$$\bar{\sigma}(t) = F[\bar{\epsilon}(t)] - \int_0^t G[\bar{\sigma}(t-\nu)] \bar{\epsilon}'(\nu) d\nu \quad (\text{III.22})$$

on the consideration that the acting stresses or rates of strain are variable within the interval of  $(0, t)$ . The equation of state can now be expressed as the equation of solid (Fig.5):

$$\bar{\sigma} = \bar{F}(\bar{\epsilon}) \quad (\text{III.23})$$

or the equation of liquid

$$\dot{\bar{\epsilon}} = \bar{G}(\bar{\sigma}). \quad (\text{III.24})$$

In the subsequent procedure we shall start from the equation of solid in which equations (III.21) and (III.22) are introduced, i.e.

$$F[\bar{\epsilon}(t)] - \int_0^t G[\bar{\sigma}(t-\nu)] \bar{\epsilon}'(\nu) d\nu = \bar{F} \left[ \int_0^t F[\bar{\epsilon}(t-\nu)] \bar{\sigma}'(\nu) d\nu \right] \quad (\text{III.25})$$

and change it to a linear form by again effecting on the right-hand side of (III.25) expansion in Taylor's series and considering its first two terms only

$$\bar{\sigma} = \bar{F}_{\bar{\epsilon}=0} + \left( \frac{\partial \bar{F}}{\partial \bar{\epsilon}} \right)_{\bar{\epsilon}=0} \bar{\epsilon} \quad (\text{III.25})$$

Quantity  $\bar{F}_{\bar{\epsilon}=0}$  again expresses the inhomogeneity of initial stresses and will be considered zero. The second term gives the initial elastic stress and the third, the stress ensuing from the stress history. In case of linear properties, equations (III.21) and (III.22) assume the form of

$$\bar{\epsilon}(t) = \int_0^t \dot{\epsilon}(t-\nu) \sigma(\nu) d\nu, \quad (\text{III.27})$$

$$\bar{\sigma}(t) = F'_{\bar{\epsilon}=0} \bar{\epsilon} - \int_0^t \dot{\sigma}(t-\nu) \dot{\epsilon}(\nu) d\nu \quad (\text{III.28})$$

and the equation of a viscoelastic substance of linear properties is obtained from (III.25) on introducing in it according to (III.27) and (III.28)

$$F'_{\bar{\epsilon}=0} \bar{\epsilon} - \int_0^t \dot{\sigma}(t-\nu) \dot{\epsilon}(\nu) d\nu = \bar{F}'_{\bar{\epsilon}=0} \int_0^t \dot{\epsilon}(t-\nu) \sigma(\nu) d\nu. \quad (\text{III.29})$$

Integrals of the left and right-hand side of (III.29) are now successively integrated by parts

$$\int_0^t \dot{\sigma}(t-\nu) \dot{\epsilon}(\nu) d\nu = \left[ \dot{\epsilon}(\nu) \int_0^t \dot{\sigma}(t-\nu) d\nu \right]_0^t - \int_0^t \frac{\partial \dot{\epsilon}(\nu)}{\partial \nu} \int_0^t \dot{\sigma}(t-\nu) d\nu, \quad (\text{III.30})$$

$$\int_0^t \dot{\sigma}(t-\nu) \dot{\epsilon}(\nu) d\nu = \left[ \dot{\epsilon}(\nu) \int_0^t \dot{\sigma}(t-\nu) d\nu \right]_0^t - \dots - (-1)^k \left[ \frac{\partial^k \dot{\epsilon}}{\partial \nu^k} \int_0^t \dot{\sigma}(t-\nu) d\nu \right]_0^t$$

and similarly

$$\int_0^t \dot{\epsilon}(t-\nu) \sigma(\nu) d\nu = \left[ \sigma(\nu) \int_0^t \dot{\epsilon}(t-\nu) d\nu \right]_0^t - \int_0^t \frac{\partial \sigma(\nu)}{\partial \nu} \int_0^t \dot{\epsilon}(t-\nu) d\nu,$$

$$\int_0^t \epsilon(t-v) \sigma(v) dv = \left[ \sigma(v) \epsilon(t-v) \right]_0^t - (-1)^k \left| \frac{\partial^k \sigma(v)}{\partial v^k} \right| \epsilon(t-v) \Big|_0^t \quad (\text{III.31})$$

Assuming for  $0 < t < \tau$  that

$$\left| \frac{\partial^k \sigma(v)}{\partial v^k} \right| \epsilon(t-v) \Big|_0^{\tau_k} = \sigma^{(k)}(t) \int_0^{\tau_{k+1}} \epsilon(t-v) dv^k, \quad (\text{III.32})$$

$$\left| \frac{\partial^k \epsilon(t)}{\partial t^k} \right| \sigma(t-v) \Big|_0^{\tau_k} = \epsilon^{(k+1)} \int_0^{\tau_{k+1}} \sigma(t-v) dv^k,$$

introducing moreover for small increments

$$\frac{d^k \sigma(v)}{dv^k} \approx \sigma^{(k)}(v), \quad \frac{d^k \epsilon(v)}{dv^k} \approx \epsilon^{(k)}(v) \quad (\text{III.33})$$

and denoting further

$$P_k = (-1)^k \int_0^{\tau_{k+1}} \sigma(t-v) dv^k, \quad (\text{III.34})$$

$$R_k = (-1)^{k-1} \int_0^{\tau_k} \epsilon(t-v) dv^k,$$

we obtain by introducing (III.30) and (III.31) in (III.29),  
equation

$$F'_{\epsilon=0} \epsilon + P'_1 \epsilon + \dots + P'_k \epsilon^{(k)} = F'_{\epsilon=0} (\sigma R_1 + \sigma R_2 + \dots + \sigma R_{k+1}) \quad (\text{III.35})$$

which corresponds to Alfrey's equation (II.5). The advantage of

the method used lies in the fact that the coefficients of the equation are already determined. For small increments  $\epsilon(t)$  we may also take  $\bar{F}_{\epsilon=0}$  and  $\bar{F}'_{\epsilon=0}$  as equal and introduce for them in (III.35). The determination of constants and functions used above will be dealt with in the section that follows.

IV Rheologic constants of viscoelastic substances

When evolving the equation of viscoelastic solids and viscoelastic liquids we have based our considerations on functional relationships that may be obtained in basic experiments- tensile test and viscosity test.

To determine the relaxation times of effects higher than the rate of strain  $\dot{\epsilon}$  or the rate of stress  $\dot{\sigma}$  (for which the first relaxation time  $\tau_1$  is the fundamental experimental datum) we shall use equations (III.8) and (III.16) and with their aid express the relaxation time

$$\tau_{m,\epsilon}^m = \frac{\left(\frac{d^m \epsilon}{dt^m}\right)_{t=0} \frac{t^m}{m!}}{\epsilon(t)} \quad (IV.1)$$

or

$$\tau_{m,\sigma}^m = \frac{\left(\frac{d^m \sigma}{dt^m}\right)_{t=0} \frac{t^m}{m!}}{\sigma} \quad (IV.2)$$

Quantities on the right-hand side of equations (IV.1) and (IV.2) may be ascertained from experimental curves of  $\epsilon(t)$  and  $\sigma(t)$  if we also plot for them the curves of  $\left(\frac{d\epsilon}{dt}\right)$ ,  $\left(\frac{d^2\epsilon}{dt^2}\right)$ , .....  $\left(\frac{d\sigma}{dt}\right)$ ,  $\left(\frac{d^2\sigma}{dt^2}\right)$ , ..... etc. (Fig.6). The right-hand sides of equations (IV.1) and (IV.2) now define the relaxation time as a ratio of two functions which is constant within a

certain time interval. This means that the two functions must be affine. Since the numerator contains function  $t^m$  and the denominator an experimental quantity, the equations will not be satisfied unequivocally.

For this reason we must approximate curves  $\left(\frac{d^m \epsilon}{dt^m}\right) \approx \epsilon^{(m)}(t)$  by always affine functions  $t^m$ . The procedure is clearly evident from Fig. 6. in which the first and second relaxation times of a chosen experimental curve for a viscoelastic solid are established. Curves indicating the course of higher derivatives are obtained either graphically or numerically from the fundamental curves of  $\epsilon(t)$  and  $\sigma(t)$ .

Functions  $\sigma(t-\nu)$  and  $\epsilon(t-\nu)$  also termed "memory" functions are other quantities that describe the properties of viscoelastic substances. The functions give the values of strain and stress in a substance in time  $t$  due to unit impulses of stress  $\delta(\sigma)$  and strain  $\delta(\epsilon)$  acting on the material in time  $(t-\nu)$ . They are established from curves of  $\sigma(t)$  and  $\epsilon(t)$  with the aid of the following relations (Fig. 7):

$$\sigma(t-\nu) = \frac{\sigma(t-\nu)}{\sigma_0}, \quad \epsilon(t-\nu) = \frac{\epsilon(t-\nu)}{\epsilon_0} \quad (\text{IV.3})$$

Analytically expressed functions are then introduced in integrals (III.35) and the integration carried out as indicated.

The fact that all coefficients of equations (III.8), (III.16), and (III.35) are determined unequivocally from experimental curves, provides the necessary prerequisites for elucidation of their physical meaning.



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